Mathematical Physics & High-Dimensional Systems

Tuesday, Sept. 1: 11:00 – 13:00

Session 2: Hilbert Hall

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Hyperbolicity and the effective dimension of spatially-extended dissipative systems

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Using covariant Lyapunov vectors, we reveal a split of the tangent space of standard models of one-dimensional dissipative spatiotemporal chaos: a finite extensive set of N dynamically entangled vectors with frequent common tangencies describes all the physically relevant dynamics and is hyperbolically separated from possibly infinitely many isolated modes representing trivial, exponentially decaying perturbations. We argue that N can be interpreted as the number of effective degrees of freedom, which has to be taken into account in numerical integration and control issues.

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Generalized Lyapunov exponent toward a unified view of dynamical instabilities

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In statistical mechanics, chaos plays an important role in deriving a stochastic description from the microscopic dynamics [1]. Ergodicity, i.e., the time average of an observation function equals to its space average, guarantees an equilibrium state in dynamical systems. Because macroscopic quantities, which result from the time average of the microscopic observation function, are almost constant if the microscopic dynamics are ergodic. On the other hand, macroscopic observables are not constant in non-equilibrium state. In other words, the time average of observation function in underlying microscopic dynamics needs to be intrinsically random in non-equilibrium state. However, the foundation of non-equilibrium statistical mechanics on the basis of the time average has not been studied at all.

Infinite measure systems, dynamical systems with an infinite invariant measure, have been attracted much attention to found the non-equilibrium statistical mechanics [2–6]. The remarkable point in infinite measure systems is that the time average of some observation function becomes intrinsically random [4,7]. To be more precise, the time average of the observation function converges in distribution, and its distribution depends on the class of the observation function. This reminds us of a randomness of the time average in non-equilibrium states.

We have studied the subexponential instability of one-dimensional maps to investigate the non-equilibrium statistical mechanics on the basis of the time average. We show that one-dimensional maps with the subexponential instability have an infinite invariant measure, where the subexponential instability is characterized as the average of the logarithm of the separation $\langle \ln \Delta x(n)/\Delta x(0)\rangle$. That is, in the subexponential instability, there exists the sequence $a_n = o(n)$ such that

$$\left\langle \frac{1}{a_n}\sum_{k=0}^{n} \ln |T'(x_k)| \right\rangle \to 1 \quad \text{as } n \to \infty,$$

where $\langle\cdot\rangle$ represents the average with respect to an initial ensemble.[1]

To characterize the subexponential instability, we propose the generalized Lyapunov exponent for a one-dimensional map as

$$\lambda(a_n) \equiv \left\langle \frac{1}{a_n}\sum_{k=0}^{n-1} \ln |T'(T^k x)| \right\rangle,$$

[1] Thanks to Aaronson's theorem [7], one can choose the arbitrary initial ensemble which is absolutely continuous with respect to the Lebesgue measure

where a_n is a monotonically increasing sequence. As a result, chaos is classified into three different classes: extremely strong chaos, chaos, non-stationary chaos. As for $a_n \sim n^\alpha L(n)$,[2] we classify chaos as follows:

1. Extremely strong chaos : $\lambda(n) = \infty$ and $\lambda(a_n) < \infty$ for $\alpha \geq 1$.
2. Chaos : $0 < \lambda(n) < \infty$.
3. Non-stationary chaos : $\lambda(n) = 0$ and $\lambda(a_n) > 0$ for some $\alpha \leq 1$.

We call $(a_n, \lambda(a_n))$ *Lyapunov pair* if there exists a_n such that $0 < \lambda(a_n) < \infty$. We will show extremely strong chaos, chaos and non-stationary chaos using one-dimensional maps.

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[2] $L(n)$ is slowly varying at ∞.

Estimating the Lyapunov Spectrum from Bred vectors in spatio-temporal chaos

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We propose a method to estimate the spectrum of Lyapunov exponents corresponding to the most expanding directions using bred vectors. Bred vectors are stationary finite fluctuations which are periodically normalized. The most frequent application of bred vectors are ensemble forecasts for weather predictions, since their computation is less demanding than the computation of Lyapunov vectors.

It has been demonstrated that the spatio-temporal dynamics of perturbations in spatially extended chaotic systems can be related to properties of scale invariant growing surfaces [2–4]. We study now, in the model proposed by Lorenz in 1996 [5], whether similar scaling properties, can also be observed for bred vectors. Therefore we use only one bred vector with constant amplitude ϵ_0 [9] instead of an ensemble of bred vectors and we vary the magnitude of the perturbation by varying ϵ_0. This method is in more detail described in [6]. If ϵ_0 is very small, the corresponding bred vector is effectively the leading Lyapunov vector. As ϵ_0 increases, the size of the perturbation increases and the corresponding perturbed trajectory differs significantly from the unperturbed trajectory.

In analogy to Lyapunov exponents, we can now compute "*bred exponents*" $\lambda_m^{BV} = \frac{1}{\Delta t}\langle \ln \frac{|\delta \mathbf{u}_m(x,t+\Delta t)|}{|\delta \mathbf{u}(x,t)|}\rangle$ in order to estimate the Lyapunov spectrum. Note however, that the index m of the bred vectors does not correspond to the index of Lyapunov vectors. More precisely, the index m of the bred vector is not a discrete, but a continuous index, given by the logarithm of the perturbation amplitude ϵ_0. In order to estimate the spectrum of Lyapunov exponents we therefore have to find a mapping $f : m \to n$, which relates a bred vector with a given logarithmic perturbation amplitude m to the n-th Lyapunov vector.

We obtain such a mapping by constructing a surface $h_m(x,t)$ via a Hopf-Cole transformation

$$h_m(x,t) = \ln |\delta \mathbf{u}_m(x,t)|, \quad \text{with} \quad \delta \mathbf{u}_m(t) = [\delta u_m(x,t)]_{x=1}^{x=L}. \tag{3.1}$$

The surface growth formalism allows to identify different universality classes in spatio-temporal chaotic systems [2]. Especially the universality class of KPZ (Karda-Parisi-Zhang) has been widely observed in non-Hamiltonian systems. This holds also for the Lorenz '96 system studied in this contribution [4]. Consequently, the structure factors $S(k) = \langle \hat{h}_m(k,t)\hat{h}_m(-k,t)\rangle_t$, with $\hat{h}_m(k,t) = \sum_x \exp(ikx) h_m(x,t)$ decay as k^{-2} as $\epsilon_0 \to 0$. For larger values of the perturbation strength ϵ_0 and small k the structure factors decays significantly slower than k^{-2} up to a certain value $k_c(m)$. Thus, the values $k_c(m)$, indicate the crossover between the two different regimes, (flat and k^{-2}) and

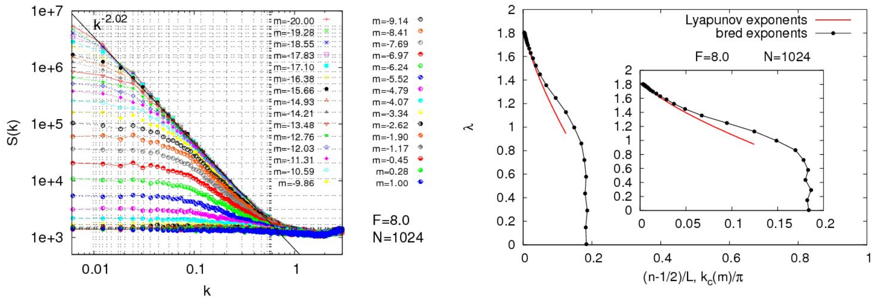

Fig. 3.1: The structure factors $S(k)$ for a Lorenz '96 system of length $L = 1024$ and $F = 8.0$ and varying perturbation amplitudes (left) and the obtained estimate of the Lyapunov spectrum (right).

represent cut off lengths l_c. Frequencies larger than $k_c(m)$ correspond to KPZ-like structures of sizes smaller than l_c.

We can use the estimated values $k_c(m)$ in order to relate Lyapunov and bred exponents via their corresponding cut of lengths $l_c^{LV} \approx (L/n)^{\theta}, \quad \text{with} \quad \theta \approx 1$ and $l_c^{BV} = \frac{\pi}{k_c(m)}$. We then identify corresponding exponents by identifying corresponding length scales,

$$\lambda_n^{LV} = \lambda_m^{BV}, \quad \text{if} \quad l_c^{LV}(n) = l_c^{BV}(m) \tag{3.2}$$

As Fig.(3.1) indicates, this method works well for the first part of the spectrum, close to the leading Lyapunov exponent, since the bred vectors have a piecewise KPZ structure in this regime.

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Low-dimensional Dynamics from Spatio-Temporal Data: An Approach Based on Reproducing Kernel Methods

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We propose an approach for re-modeling the time evolution law that describes low-dimensional dynamics in a high-dimensional state space from simulation or experimental data. The key of proposed approach is the use of the kernel methods [1] recently developed in the filed of machine learning. The problem is formulated as the finding of optimal nonlinear transformations $\phi(\cdot)$ and $\psi(\cdot)$ such that the correlation coefficient between $\phi(\mathbf{x}(t))$ and $\psi(\mathbf{x}(t+\Delta t))$, where $\mathbf{x}(t)$ and $\mathbf{x}(t+\Delta t)$ are two adjacent state points of the system, is maximized from some class of functions (a reproducing kernel Hilbert space: RKHS). This optimization problem is solved using the kernel canonical correlation analysis (kernel CCA) [2] which is a version of the kernel methods. It has already been successfully applied to various problems of pattern recognition, bioinformatics, and brain science. Recently, it has been also shown that kernel CCA is a powerful tool for analyzing complex nonlinear dynamics [3]. We present several examples of ordinary and partial differential equations which show that the proposed method is useful for analyzing low-dimensional motions of high-dimensional dynamical systems.

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Scaling of singular vectors and finite-time Lyapunov exponents in spatiotemporal chaos

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Singular vectors (SVs) are currently used in forecasting with operative models, and are useful in other contexts. For any given dynamical system, an infinitesimal perturbation evolves linearly: $\delta\mathbf{u}(t+\tau) = \mathbf{M}(t+\tau,t)\delta\mathbf{u}(t)$. The SV is defined as the perturbation at time t that gets amplified the most at $t+\tau$, and it satisfies the eigenvalue problem $\mathbf{M}^*(t+\tau,t)\mathbf{M}(t+\tau,t)\mathbf{s}_\tau(t) = \mu_\tau(t)\mathbf{s}_\tau(t)$.

We have studied the SV in two prototypical systems exhibiting spatiotemporal chaos [1]: A one-dimensional coupled-map lattice (CML) with logistic maps $f(\varrho) = 4\varrho(1-\varrho)$, and the Lorenz 96 model (L96). It is convenient for our theory to define a "surface" associated with the SV through a logarithmic transformation:

$$h_\tau(x,t) = \ln|s_\tau(x,t)| \tag{5.1}$$

Figure 5.1(a) shows that, if τ is not too large, h_τ exhibits a triangular structure what implies an exponential localization of the SV.

We have found a minimal Langevin model that allows to understand in the simplest terms the structure of the SV surface. It is a modification of the KPZ equation [2] considering a time-periodic noise (PNKPZ):

$$\partial_t h_\tau(x,t) = \zeta_\tau(x,t) + [\partial_x h_\tau(x,t)]^2 + \partial_{xx} h_\tau(x,t), \tag{5.2}$$

where one simply assumes ζ_τ to be "white noise" with period τ. The solutions of (5.2) exhibit a triangular structure like the SV surfaces, see Fig. 5.1(b).

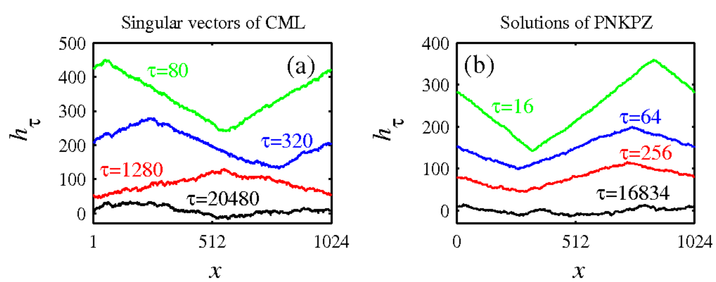

Fig. 5.1: (a) SV surfaces of the CML model for different values of τ. (b) Asymptotic solutions of the PNKPZ equation for different values of τ.

In both, chaotic systems and PNKPZ, the surface width $W_\tau^2(L) = L^{-1}\sum_x h_\tau(x,t)^2 - [L^{-1}\sum_x h_\tau(x,t)]^2$, satisfies the same scaling:

$$W_\tau^2 \sim \begin{cases} L^{2\alpha_{QC}}\tau^{-\gamma} & (\tau \ll \tau_\times(L)) \\ L^{2\alpha_{KPZ}} & (\tau \gg \tau_\times(L)) \end{cases} \tag{5.3}$$

Assuming $\tau_\times(L) \sim L^{z_{KPZ}}$, we obtain a formula for the exponent γ:

$$\gamma = \frac{2(\alpha_{\rm QC} - \alpha_{\rm KPZ})}{z_{\rm KPZ}}. \tag{5.4}$$

Inserting the critical exponents in one dimension —$\alpha_{\rm KPZ} = 1/2$ and $z_{\rm KPZ} = 3/2$ from KPZ [2]; and $\alpha_{\rm QC} = 1.07 \pm 0.05$ from columnar KPZ [3]— we get $\gamma = 0.76 \pm 0.07$ in good agreement with the result of our simulations ($\gamma \simeq 0.78$).

It is well known that as the the time horizon τ increases the SV approaches the (forward) Lyapunov vector. Then, the finite-time Lyapunov exponent (LE) —defined as $\lambda_\tau(t) = (2\tau)^{-1} \ln \mu_\tau(t)$— approaches the LE: $\lim_{\tau\to\infty} \lambda_\tau(t) = \lambda$. We have found that (see Fig. 5.2):

$$(\lambda_\tau - \lambda) \sim \begin{cases} \tau^{-\gamma} & (\tau \ll \tau_\times) \quad \textbf{our result} \\ \tau^{-1} & (\tau \gg \tau_\times) \quad \text{known asymptotics} \end{cases}$$

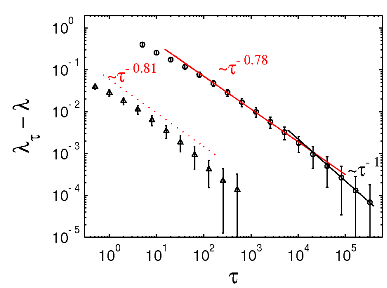

Fig. 5.2: Finite-τ deviation of the Lyapunov exponent for CML ($\circ$) and L96 ($\triangle$).

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Blow-up analysis of glycolytic relaxation oscillations

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An interesting feature of many biological processes is that they evolve on different time scales. Such processes can be modeled by singularly perturbed systems of ordinary differential equations. Classically such problems have been analyzed by the method of matched asymptotic expansions. A more recent powerful approach to singularly perturbed problems, based on methods from dynamical systems theory, has become known as geometric singular perturbation theory, see e.g. [2], [4].

In this talk a singularly perturbed planar system modeling oscillatory patterns in glycolysis, so-called glycolytic oscillations, is analyzed. Namely, the following system is considered

$$\begin{aligned} \dot{a} &= a^2b^2(\mu - 1) + \mu\delta^2, \\ \varepsilon\dot{b} &= a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2), \end{aligned} \tag{6.1}$$

where ε plays the role of a singular perturbation parameter and δ is viewed as a small parameter. The parameter μ does not play an important role in the analysis, hence we fix it and consider problem (6.1) as a two-parameter problem with the parameters ε and δ.

System (6.1) with ε small is in the standard form of slow-fast systems with the slow variable a and the fast variable b. For certain parameter values the system exhibits a stable limit cycle of relaxation type. For $\varepsilon = 0$ the slow dynamics is restricted to a critical manifold S defined as

$$S = \{(a,b)|\ a^2b^2(1-b) + \delta^2(a^2b^2 - b + \delta^2) = 0\}.$$

For $\delta > 0$ this critical manifold is an N-shaped curve leading to the relaxation oscillations, i.e. in the limit $\varepsilon \to 0$ for fixed $\delta > 0$ the situation is essentially as in the classical Van der Pol oscillator.

However, the limit $\varepsilon \to 0$ for δ fixed is highly non-uniform with respect to δ, i.e. the shape of the critical manifold S is greatly affected by δ. In particular, for $\delta = 0$ the N-shaped manifold S collapses into a more singular set defined by $a^2b^2(1-b) = 0$ which consists of the lines $a = 0$, $b = 0$ and $b = 1$. This shows that δ affects the geometry of S, while ε plays the role of a singular perturbation parameter causing the slow-fast structure. These features motivated much of our interest in the analysis of the case where ε and δ tend to zero simultaneously.

In a previous work on glycolytic oscillations Segel and Goldbeter [6] applied the method of scaling to explain the occurrence of the relaxation oscillations. These authors pointed out that this method was based on the thorough understanding of the underlying phenomenon. They pose a problem to find a systematic method based on the mathematical structure of the equations. In this talk such a method will be presented.

Our geometric singular perturbation approach to the limit $(\varepsilon, \delta) \to (0,0)$ complements the study presented in [6] and provides a rigorous analysis of relaxation oscillations. In particular, the blow-up method pioneered by Dumortier and Roussarie [1] in their geometric analysis of canards in the Van der Pol equations has proven to be very useful in the analysis of singularly perturbed problems with the degenerate critical manifolds [2], [3], [5].

It turns out that two blow-ups of the $\delta = 0$ degenerate critical manifold with respect to δ lead to a complete desingularization of the problem such that uniform results in ε become possible. In this approach the degenerate lines $a = 0$ and $b = 0$ are blown-up to cylinders by rewriting the original (a, b, δ) variables in suitable cylindrical variables. In the blown-up geometry the scaling regimes of Segel and Goldbeter are recovered. In addition, a rigorous asymptotic matching of these regimes becomes possible.

This is joint work with Peter Szmolyan (Vienna University of Technology).

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Modelling of high-dimension dynamics by random dynamical systems

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Abstract: The majority of natural systems are known to be open, i.e., subject to numerous external forcings; these forcings can be modeled by random dynamical systems (RDS). The RDS present a necessary and important step towards reconstructing the observed systems when their adequate first-principle mathematical models are either unknown or subjected to further verification. Note that, even for deterministic systems, the construction of a deterministic model from the observed TS and use of this model for prediction has quite a number of principal restrictions.First, according to the Takens theorem, the reconstruction of a phase trajectory is possible in a phase space of sufficiently high dimension $d_E > 2d_S + 1$, where d_S is the phase space dimension of the system that has generated the initial TS. This means that a deterministic dynamical system (DDS) describes correctly behavior of the reconstructed system in the subspace of dimension d_S that is much smaller than the dimension of the phase space d_E of the model. Consequently, the model is not adequate for the system at relatively small changes of control parameters. The second restriction is the limitation on prior information. To confirm determinism of the observed system one has to ensure that the attractor reconstructed by the TS has a finite dimension and to find the smallest embedding dimension for this attractor. The available methods of determining such dimensions are inapplicable for analysis of the TS generated by real systems. Reconstruction in the form of RDS (stochastic model) removes these restrictions, thus making the proposed approach more universal. A basic idea underlying the stochastic approach is that the robust dynamic properties of the system evolution can be described by a few equations, while other features may be considered as a stochastic disturbance. A principal new step here is inclusion of parameterized stochastic perturbation in the model of the evolution operator; it allows us to significantly expand a class of reconstructed systems. The method of parameterization of such models on the basis of artificial neural networks is developed, as well as technique of investigation of model parameter space is suggested. Possibilities of the approach with reference to the analysis of time series generated by high dimensional dynamic systems are demonstrated by model examples. In particular, the prediction of changes of characteristics of observed process is constructed. Possible other applications of the method are discussed.

Partially integrable dynamics of populations of nonidentical oscillators with global nonlinear coupling

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We consider oscillator ensembles consisting of subpopulations of identical units, with a general heterogeneous coupling between subpopulations. Using the Watanabe-Strogatz (WS) ansatz [1], we reduce the dynamics of the ensemble to a relatively small number of dynamical variables plus constants of motion [2]. This reduction is independent of the sizes of subpopulations and remains valid in the thermodynamic limits. First we apply the theory to the standard Kuramoto model and make a link to the recent Ott–Antonsen (OA) ansatz [3]. Next, we use it to describe two interacting subpopulations, where we generalize the results of [4] and report a novel, quasiperiodic chimera state. Finally, we use the theory to analyze dynamics of ensembles with global nonlinear coupling [5, 6]; the main finding here is multistability of the mean field dynamics.

Our basic model is a generalization of the Kuramoto model: we consider the ensemble as a hierarchical structure, consisting of subpopulations of identical units, with a homogeneous coupling within a subpopulation and generally a heterogeneous coupling between subpopulations. Labelling the subpopulation with indices a, b we write for the phase of the kth oscillator of the subpopulation a:

$$\frac{d\phi_k^a}{dt} = \omega_a + \mathrm{Im}(H_a e^{-i\phi_k^a}) ,$$

where H_a is the effective force acting on the oscillators of subpopulation a:

$$H_a = \sum_b n_b E_{a,b} r_b e^{i\Theta_b} .$$

Here complex parameter $E_{a,b}$ describes the coupling between subpopulations a, b, $n_a = N_a/N$ are relative population sizes, and $r_a e^{i\Theta_a} = N_a^{-1}\sum_{k=1}^{N_a} e^{i\phi_k^a}$ is the complex mean field of a-th subpopulation.

We apply the WS ansatz to *each subpopulation* and in this way reduce its dynamics to that of three variables plus $N_a - 3$ constants of motion. Next we perform a thermodynamic limit $N \to \infty$; there are two main ways to do this. (i) The number of subpopulations M remains finite, but their sizes grow $N, N_a \to \infty$ in a way that $n_a = const$. In this limit the ensemble is described by a set of $3M$ ODEs. (ii) In another limiting case, we keep the size of each subpopulation N_a finite but let the number of subpopulations grow, $M \to \infty$. In this way we describe a population with a continuous frequency distribution.

Applying the thermodynamic limit (ii) to the standard Kuramoto model we derive as a particular case the OA equation [3] for the evolution of the mean field. This equation was obtained in [3] under an assumption of a certain parametrization of the initial distribution of phases. We demonstrate that this assumption corresponds to the case of the uniform distribution of constants of motion in the WS theory. Applying the thermodynamic limit (ii) to the system of two interacting subpopulations, considered in [4], we obtain a full description of the system. The equations of [4], based on the OA assumption, are shown to be a particular case of our equations, which exhibit also novel solutions, corresponding to quasiperiodic chimera states.

Finally, we apply our formalism to an ensemble with a continuous frequency distribution and homogeneous coupling $E_{a,b} = E$. The latter can, however, depend on the amplitude r of the mean field, $E = \varepsilon A(r)e^{i\beta(r)}$ (global nonlinear coupling, see [5, 6]). For an interesting case of phase nonlinearity $A = 1$, $\beta = \beta_0 + \varepsilon^2 r^2$ we obtain that (i) mean field amplitude is not a monotonically increasing function of ε and (ii) collective dynamics becomes mulistable: there can coexist several synchronous solutions with different amplitudes and frequencies of the mean field. Theoretical results are supported by direct numerical simulation of ensemble dynamics.

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How well can one resolve the state space of a chaotic map with noise?

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Any physical system suffers background noise, any numerical prediction suffers from computational roundoff errors, and any set of equations models nature up to a given accuracy, since degrees of freedom are always neglected. The effect of noise on the behavior of a nonlinear dynamical system is a fundamental problem in many areas of scienceand the interplay of noise and chaotic dynamics is of particular current interest The main purpose of our work is to study an effect of noise that has not been addressed in literature: weak noise limits the attainable resolution of the state space ('phase space') of a chaotic system. Our task is to develop a method to estimate the finest possible partition of the state space, which depends on the nontrivial interplay between the stretching/contraction of the deterministic dynamics and the smearing effect of noise. As the optimal partition is finite, Fokker-Planck equations can be represented by a finite matrix, whose leading eigenvalue gives a good estimate of such long-time observables as escape rates, Lyapunov exponents, *etc*...

The method is here illustrated for a $1d$ map $x_{n+1} = f(x_n) + \xi_n$, where the ξ_n are independent Gaussian random variables of mean 0 and variance $2D$. We follow a 'Fokker-Planck'-type of approach, and work with densities of trajectories, whose discrete-time dynamics is determined *forward* by the Fokker-Planck operator $\mathcal{L} \circ \rho_n(y) = \int \frac{dx}{\sqrt{4\pi D}}\, e^{-\frac{(y-f(x))^2}{4D}} \rho_n(x)$ and *backward* by its adjoint.

The set of unstable periodic orbits is the 'skeleton' of the deterministic dynamics, and therefore it can be used to partition the state space into a set of regions, each region a neighborhood of an unstable periodic point. The number of periodic orbits grows exponentially with period length, yielding finer and finer partitions, with the neighborhood of each periodic orbit shrinking exponentially. In the presence of weak noise, we switch to the Fokker-Planck picture and partition the state space using eigenfunctions (as 'noisy periodic orbits') of the adjoint Fokker-Planck operator, linearized in the neighborhood of periodic points of the deterministic map. Such eigenfunctions are Gaussians, whose variances represent the balance between noise and deterministic dynamics. Now, *'the best possible of all partitions'* is determined by the following algorithm: assign to each periodic point x_a a neighborhood of finite width $[x_a - \sigma_a, x_a + \sigma_a]$. Consider periodic orbits of increasing period n_p, and *stop the process of refining* the state space partition as soon as the adjacent neighborhoods overlap. In the example shown in figure 9.1, evolution in one time-step of all the regions of the optimal partition is compactly summarized by a transition graph.

Periodic orbit theory [2] expresses the long-time dynamical averages, such as Lyapunov exponents, escape rates, and correlations, in terms of the leading

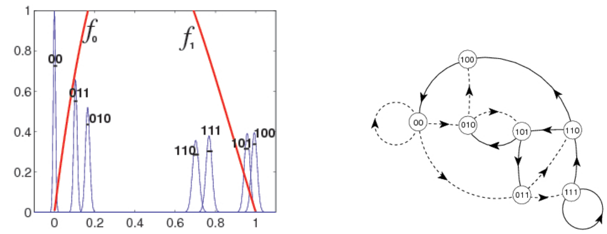

Fig. 9.1: (left) optimal partition of a cubic repeller for 2D=0.002; (right) corresponding transition graph.

eigenvalues of the Fokker-Planck operator $\mathcal{L}$. By means of the optimal partition, we now have a finite number of periodic orbits, which the periodic orbit expansion is supported on. The escape rate of the map is now the leading root of the determinant of the graph [1]. We then compute the escape rate of the same map using (a) several deterministic, over-resolved partitions, and (b) a brute force numerical discretization of the Fokker-Planck operator, which we take as our reference value. It turns out that the escape rate estimated by means of the optimal partition is consistent with what found with the brute force discretization. On the contrary, successive estimates of the escape by over-resolved expansions appear to converge to a value significantly different from the 'optimal partition' one. We then investigate the range of validity of the optimal partition method with respect to the noise amplitude, and find it in agreement of at least 2% with the brute force discretization.

Future work includes the extension of the optimal partition method to non-hyperbolic one-dimensional maps, higher dimensional maps and flows.

References

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