Holmogorov phenomenology (see Pope 2000, Frisch 1955)

Exact statistical evolution equations are generally unclosed
so exact solution out of reach
so phenomenological assumptions needed

Holmogorov 1941

Holmogorov's hypotheses:

1) Local isotropy: At sufficiently high Reynolds number, the small-scale turbulent motions are statistically isotropic.

2) First similarity hypothesis: In every turbulent flow at sufficiently high Reynolds number, the statistics of the small-scale motions have a universal form that is uniquely determined by $\nu$ and $\langle \varepsilon \rangle$.

3) Second similarity hypothesis: In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions in the inertial range have a universal form that is uniquely determined by $\langle \varepsilon \rangle$, independent of $\nu$.

Goal: determine statistical features of the flow under these hypotheses

Holmogorov scales: On small scales, $\nu$ and $\langle \varepsilon \rangle$ are the relevant quantities:

$$\langle \varepsilon \rangle = \frac{\nu^2}{S^2}, \quad [\nu] = \frac{\nu^2}{S}$$

Co length scale $l = \left( \frac{\nu^2}{\langle \varepsilon \rangle} \right)^{1/4}$

Time scale $\tau_n = \left( \frac{\nu}{\langle \varepsilon \rangle} \right)^{1/2}$
velocity scale \( \nu = \frac{u_2}{\nu} = (\nu \langle \varepsilon \rangle)^{1/4} \)

- Reynolds number on the Kolmogorov scale \( Re^* = \frac{u_2 \nu}{\nu} = 1 \)

flow is laminar on the Kolmogorov scale

Kolmogorov scales indeed characterize the small scales!

Energy spectrum according to \( k41 \)

\[ E(k) \]

- large (integral) scales:
  - statistics non-universal
  - depend on large-scale forcing

- inertial range:
  - statistics universal
  - depend on \( \langle \varepsilon \rangle \)

- dissipative range:
  - statistics universal
  - depend on \( \langle \varepsilon \rangle \) and \( \nu \)

\[ [E(k) \, dk] = \frac{4 \nu^2}{5^2} \, C_0 \quad [E(k)] = \frac{4 \nu^3}{5^2} \]

\[ E(k) = \overline{F(k, \varepsilon)} \sim \varepsilon^{\alpha} k^\beta \]

\[ \alpha = \frac{3}{5} \quad \beta = -\frac{5}{3} \]

\[ E(k) = C_k \varepsilon^{2/3} k^{-5/3} \]

\( k41 \) predicts \( k^{-5/3} \) inertial range scaling

\[ C_0 \quad \text{homework!} \]

\[ \text{Helmholtz constant } O(1) \]
dissipative range: if velocity field is smooth, energy spectrum needs to decay faster than algebraic:

argument in 1D:

\[ R(r) = \langle u(x)u(x+r) \rangle = \int dk \ E(k) e^{ikr} \]

\[ C_0 \langle \partial_x^2 u \rangle^2 = \lim_{r \to 0} (-1)^n \int_r^\infty \langle u(x+r)u(x) \rangle \ e^{ikr} \]

\[ = \lim_{r \to 0} (-1)^n \int_r^\infty \ E(k) (ik)^{2n} \ e^{ikr} \]

\[ = \int \ E(k) \frac{1}{2n} \leq \infty \]

integral range is expected to be non-universal, behaviour for smaller depends on the Loitsianski: integral

relation to decay of kinetic energy

**Longitudinal structure functions**

\[ S_n(r) = \langle \left( \left[ u(x+r) - u(x) \right]^n \cdot \ell^2 \right)^n \rangle \]

1.4 dimensional analysis:

\[ \left[ S_2(r) \right] = \frac{\mu^2}{S^2} \]

\[ C_0 \ S_2(r) = \mathcal{F}(r, \langle \varepsilon \rangle) \sim \langle \varepsilon \rangle^{2/3} r^{2/3} \]

\[ \left[ S_3(r) \right] = \frac{\mu^3}{S^3} \]

\[ C_0 \ S_3(r) \sim \langle \varepsilon \rangle r \quad \text{consistent with H5 law!} \]

2. higher-order moments \[ S_n(r) \sim \langle \varepsilon \rangle^{n/3} \]

3. experimental observation: 1.4 accurately represents low-order statistics
velocity increment PDF (see Argns, Faust, Haase, Friedrich (2010))

- in the following, $\nu$ denotes the sample-space variable of the velocity increment

- according to 441, $f(\nu; r)$ should have a universal form in the inertial & dissipative ranges, depending on $\langle \varepsilon \rangle$ and $\nu$ only:

\[
f(\nu; r) = F(\nu, r, \langle \varepsilon \rangle, \nu) \quad \text{for} \quad r \ll L
\]

- non-dimensionalization:

\[
\bar{\nu} = \frac{\nu}{\nu_c} = \left( \frac{\nu}{\langle \varepsilon \nu \rangle} \right)^{1/4}
\]

\[
\bar{r} = \frac{r}{\nu_c} = \left( \frac{\nu^3}{\langle \nu \rangle^2} \right)^{1/4}
\]

- consequence for velocity increment PDF:

\[
f(\nu; r) = \frac{d\bar{\nu}}{d\nu} \quad G(\bar{\nu}; \bar{r})
\]

ensures normalization:

\[
\int d\nu f(\nu; r) \overset{!}{=} \int d\nu d\bar{\nu} G(\bar{\nu}; \bar{r})
\]

\[
= \int d\bar{\nu} G(\bar{\nu}; \bar{r})
\]

- in the inertial range, statistics should be independent of $\nu$:

\[
\frac{d}{d\nu} \left[ \frac{1}{\nu_c^2} G(\nu; \nu_c; \frac{\nu}{\nu_c}) \right] \overset{!}{=} 0
\]

\[
\downarrow \quad \text{homework}
\]

\[
\frac{\partial}{\partial \bar{\nu}} \left[ \bar{\nu} G(\bar{\nu}; \bar{r}) \right] + 3 \bar{r} \frac{\partial}{\partial \bar{r}} G(\bar{\nu}; \bar{r}) \overset{!}{=} 0
\]
This relation possesses the solution

\[ G(\tilde{v}, \tilde{r}) = \frac{1}{r^\beta} g\left(\frac{\tilde{v}}{r}\right) \]

441 predicts self-similar velocity increment PDFs

- i.e. when standardized, velocity increments should collapse in the inertial range
- this is not backed up by data

Intermittency!

441 is reasonable description for low-order statistics, but lacks to capture higher-order statistics.