Kolmogorov phenomenology (see Pope 2000, Frisch 1995)

- exact statistical evolution equations are generally unclosed
  ↳ exact solution out of reach
  ↳ phenomenological assumptions needed

Kolmogorov 1941

Kolmogorov's hypotheses:

1) <u>local isotropy</u>: At sufficiently high Reynolds number, the small-scale turbulent motions are statistically isotropic.

2) <u>first similarity hypothesis</u>: In every turbulent flow at sufficiently high Reynolds number, the statistics of the small-scale motions have a universal form that is uniquely determined by ν and $\langle \varepsilon \rangle$.

3) <u>second similarity hypothesis</u>: In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions in the inertial range have a universal form that is uniquely determined by $\langle \varepsilon \rangle$, independent of ν.

↳ goal: determine statistical features of the flow under these hypotheses

Kolmogorov scales: on small scales, ν and $\langle \varepsilon \rangle$ are the relevant quantities:

$[\langle \varepsilon \rangle] = \frac{m^2}{s^3}$ $[\nu] = \frac{m^2}{s}$ ↳ length scale $\eta = \left(\frac{\nu^3}{\langle \varepsilon \rangle} \right)^{1/4}$;

time scale $\tau_\eta = \left(\nu / \langle \varepsilon \rangle \right)^{1/2}$

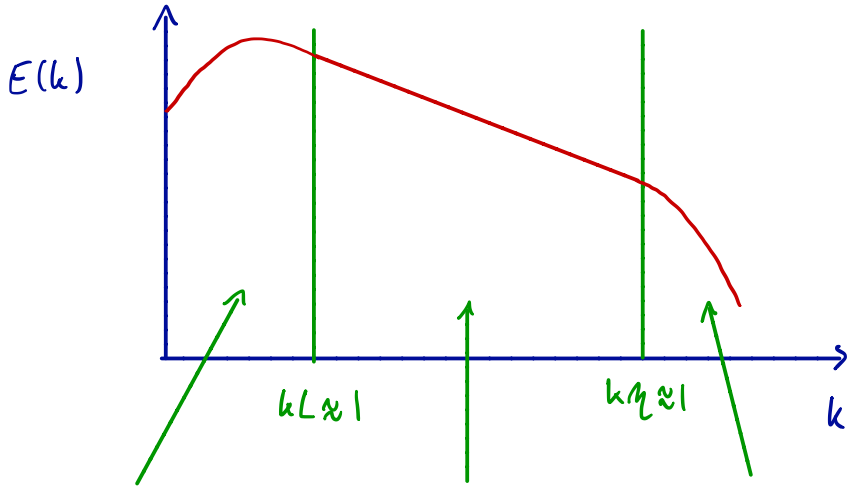
velocity scale $u_\eta = \eta / \tau_\eta = (\nu \langle \varepsilon \rangle)^{1/4}$

- Reynolds number on the Kolmogorov scale $R_\eta = \frac{u_\eta \eta}{\nu} = 1$

$\Rightarrow$ flow is laminar on the Kolmogorov scale

$\Rightarrow$ Kolmogorov scales indeed characterize the small scales!

Energy spectrum according to K41



large (integral) scales:
- statistics non-universal
- depend on large-scale forcing

inertial range:
- statistics universal
- depend on $\langle \varepsilon \rangle$

dissipative range:
- statistics universal
- depend on $\langle \varepsilon \rangle$ and ν

- inertial range: $[E(k)\,dk] = \frac{m^2}{s^2} \Rightarrow [E(k)] = \frac{m^3}{s^2}$

$\Rightarrow E(k) = F(k, \langle \varepsilon \rangle) \sim \langle \varepsilon \rangle^\alpha k^\beta \Rightarrow \alpha = 2/3 \quad \beta = -5/3$ homework!

$\Rightarrow E(k) = C_k \langle \varepsilon \rangle^{2/3} k^{-5/3}$

Kolmogorov constant $O(1)$

K41 predicts $k^{-5/3}$ inertial range scaling

- dissipative range: if velocity field is smooth, energy spectrum needs to decay faster than algebraic:

  - argument in 1D: $R(r) = \langle u(x)u(x+r)\rangle = \int dk\, E(k)\, e^{ikr}$

$$\hookrightarrow \langle (\partial_x^n u)^2 \rangle = \lim_{r\to 0} (-1)^n \partial_r^{2n} R(r)$$

$$= \lim_{r\to 0} (-1)^n \int dk\, E(k)\, (ik)^{2n} e^{ikr}$$

$$= \int dk\, k^{2n} E(k) \overset{!}{\leq} \infty$$

- integral range is expected to be non-universal, behaviour for smaller depends on the Loitsianski integral

$\hookrightarrow$ relation to decay of kinetic energy

<u>longitudinal structure functions</u>

$$S_n(r) = \left\langle \left([\underline{u}(\underline{x}+\underline{r}) - \underline{u}(\underline{x})] \cdot \hat{\underline{r}} \right)^n \right\rangle$$

- K41 dimensial analysis: $[S_2(r)] = \frac{m^2}{s^2}$

$$\hookrightarrow S_2(r) = F(r, \langle\varepsilon\rangle) \sim \langle\varepsilon\rangle^{2/3} r^{2/3}$$

$[S_3(r)] = \frac{m^3}{s^3}$ $\hookrightarrow$ $S_3(r) \sim \langle\varepsilon\rangle r$ consistent with 4/5 law!

- higher-order moments $S_n(r) \sim (\langle\varepsilon\rangle r)^{n/3}$

- experimental observation: K41 accurately represents low-order statistics

velocity increment PDF (see Argyris, Faust, Haase, Friedrich (2010))

- in the following, v denotes the sample-space variable of the velocity increment

- according to K41, $f(v;r)$ should have a universal form in the inertial & dissipative ranges, depending on $\langle\varepsilon\rangle$ and ν only:

$$f(v;r) = F(v, r, \langle\varepsilon\rangle, \nu) \quad \text{for } r \ll L$$

- non-dimensionalization: $\bar{v} = \frac{v}{u_\eta} = \frac{v}{(\varepsilon\nu)^{1/4}}$

$$\bar{r} = \frac{r}{r_\eta} = \frac{r}{(\nu^3/\langle\varepsilon\rangle)^{1/4}}$$

- consequence for velocity increment PDF:

$$f(v;r) = \frac{d\bar{v}}{dv}\, G(\bar{v};\bar{r})$$

ensures normalization: $\int dv\, f(v;r) \overset{!}{=} 1 = \int dv \frac{d\bar{v}}{dv} G(\bar{v};\bar{r})$

$$= \int d\bar{v}\, G(\bar{v};\bar{r})$$

- in the inertial range, statistics should be independent of ν:

$$\frac{d}{d\nu}\left[\frac{1}{u_\eta} G\left(\frac{v}{u_\eta}; \frac{r}{\eta}\right)\right] \overset{!}{=} 0$$

$\downarrow$ homework

$$\frac{\partial}{\partial\bar{v}}\left[\bar{v}\, G(\bar{v};\bar{r})\right] + 3\bar{r}\frac{\partial}{\partial\bar{r}} G(\bar{v};\bar{r}) \overset{!}{=} 0$$

↳ this relation possesses the solution

$$G(\bar{v};\bar{r}) = \frac{1}{\bar{r}^{1/3}}\, g\left(\frac{\bar{v}}{\bar{r}}\right)$$

↳ K41 predicts self-similar velocity increment PDFs

- i.e. when standardized, velocity increments should collapse in the inertial range
- this is not backed up by data

↳ intermittency!

↳ K41 is reasonable description for low-order statistics, but lacks to capture higher-order statistics