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## Generation of Lagrangian intermittency in turbulence by a self-similar mechanism

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**Abstract.** Intermittency, i.e., extreme fluctuations at small scales, causes the deviation of turbulence statistics from Kolmogorov's 1941 theoretical predictions. Intermittency effects are especially strong for Lagrangian statistics. Our understanding of how Lagrangian intermittency manifests, however, is still elusive. Here, we study the Lagrangian intermittency in the framework of an exact, yet unclosed probability density function (PDF) equation. Combining this theoretical approach with data from experiments and simulations, no *a priori* phenomenological assumptions about the structure or properties of the flow have to be made. In this description, the non-self-similar evolution of the velocity increment PDF is determined at all scales by a single function, which is accessible through data from experiments and simulations. This 'intermittency generating function' arises from the dependence of the acceleration of a fluid

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element on its velocity history, thereby coupling different scales of turbulent motion. Empirically, we find that the intermittency generating function has a simple, approximately self-similar form, which has the surprising implication that Lagrangian intermittency—the absence of self-similarity in the Lagrangian velocity increment statistics—is driven by a self-similar mechanism. The simple form of the intermittency generating function furthermore allows us to formulate a simple model parametrization of the velocity increment PDFs.

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## 1. Introduction

Fluid turbulence is replete with strong fluctuations, which become more and more violent as the scale decreases. In Kolmogorov's celebrated 1941 theory (K41) [1], fluctuations in the inertial range, the intermediate scales between the large scale of energy input and the smallest scale of viscous dissipation, are self-similar. The existence of extreme fluctuations in turbulence beyond the K41 prediction is termed intermittency (or internal intermittency, to distinguish from the irregular appearance of turbulent regions at the turbulent/non-turbulent interface). In fact, just after the birth of K41, Landau cast doubt on the theory by pointing to the fluctuations of the energy dissipation, the key parameter in K41 (see e.g. discussions in [2]). Kolmogorov later proposed the K62 ansatz [3] that the velocity difference between two points separated by an inertial-range distance conditioned on the instantaneous energy dissipation averaged over that scale is self-similar. This 'refined self-similarity' made it explicit, for the first time, that coupling between inertial range and dissipation scales causes intermittency, and has remained influential on the study of intermittency in the Eulerian frame, i.e. fluctuations of quantities measured at two spatial points but at a single time, even to the present day.

Intermittency is particularly apparent in the Lagrangian description of turbulence, where the spatiotemporal evolution of the flow is characterized by describing the trajectories of individual infinitesimal fluid elements. The Lagrangian approach is the natural choice for studying turbulent mixing and transport, and has seen significant theoretical, numerical and experimental advances in recent decades [4, 5]. In the Lagrangian context, intermittency is commonly investigated by studying the fluid acceleration [6–9] and its coarse-grained cousin the velocity increment [10–13]. We define the velocity increment as  $\Delta u(t, \tau) = u(t + \tau) - u(t)$ , where  $u$  is a component of the fluctuating velocity and the difference is taken along a fluid-element trajectory. It is well known that when  $\tau$  is large, the probability density function (PDF) of  $\Delta u$  is nearly Gaussian [5, 10, 11]; as  $\tau$  becomes smaller, however, the PDF rapidly becomes

highly non-Gaussian and extreme fluctuations become more and more common. This non-self-similar evolution of the PDF as a function of *scale* is a hallmark of intermittency, and must be captured by a statistical theory of turbulence.

Previous theoretical work on the Lagrangian intermittency has primarily focused on phenomenology: starting from a set of assumptions about the nature of turbulence, a model for the statistical properties of the flow is formulated from which predictions can be derived. For example, the dominant recent approach is based on multifractal analysis [14–17], which models the turbulent energy cascade as a self-similar process in scale but with a continuous spectrum of fractal dimensions. As pointed out by Chevillard *et al* [15], distinguishing inertial and dissipative scales in this modeling framework is crucial. Other approaches include models of the velocity gradient tensor [18, 19], which assume a form for the non-local contribution of the pressure; continuous-time random walk models [20], which explicitly account for time correlations of the Lagrangian acceleration; the superstatistics approach [21], which models turbulence as a superposition of stochastic processes; vortex filament calculations [22], which envision turbulence as an ensemble of vortex tubes; and a Lagrangian generalization of the classic log-Poisson model [23], which posits a particular form for the statistics of the turbulent energy dissipation.

The starting point of this work is an exact, yet unclosed equation governing the evolution of the PDF across time lags. Cast in this way, no initial phenomenological assumptions about the nature of the flow need to be made. Instead, the shape of the PDF is explicitly related to a physical quantity that can be measured in experiments and simulations. We show that the non-self-similar evolution of the PDF of the Lagrangian velocity increments is governed by a single function, which we term the *intermittency generating function*. This function includes as a key ingredient the average of the Lagrangian acceleration conditioned on the velocity increment, thus identifying the non-trivial statistical correlation of the acceleration and the velocity increment as the source of intermittency. Remarkably, we find in both numerical simulations and laboratory experiments that, unlike the PDF itself, over the inertial range of time scales, the conditional acceleration can nearly be collapsed onto a single curve. Using this observation, we formulate an analytically tractable model for the PDFs of the acceleration and the velocity increment for all scales. Our results show that a simple mechanism—that the acceleration of a fluid element is statistically dependent on its velocity history—can be used to characterize intermittency in Lagrangian turbulence in a precise way.

## 2. Probability density function (PDF) equation for Lagrangian velocity increments

We begin with the PDF  $f$  of the velocity increment, which can be expressed as an ensemble-averaged delta distribution (also known as the fine-grained PDF) as [24–26]

$$f(v; \tau) = \left\langle \delta \left( \frac{\Delta u}{\sigma} - v \right) \right\rangle, \quad (1)$$

where  $v$  is the sample-space variable for the velocity increment,  $\tau$  is the time lag over which the increment is taken,  $\delta(\cdot)$  is the Dirac delta distribution and  $\sigma(\tau) = \langle \Delta u(t, \tau)^2 \rangle^{1/2}$ . We assume statistically stationary, homogeneous and isotropic turbulence; thus,  $f$  does not depend explicitly on time  $t$  or on the initial position of the fluid element.

As described above, intermittency of the velocity increment is manifest in the non-self-similar evolution of  $f$  as a function of the time lag  $\tau$ . To study this evolution, we follow previous

related work [26–28] and differentiate  $f$  with respect to  $\tau$ , obtaining

$$\frac{\partial}{\partial \tau} f(v; \tau) = -\frac{\partial}{\partial v} \varphi(v, \tau) f(v; \tau), \quad (2)$$

where

$$\varphi(v, \tau) = \frac{\langle a|v, \tau \rangle}{\sigma(\tau)} - \frac{\langle a \Delta u \rangle(\tau)}{\sigma(\tau)^2} v \quad (3)$$

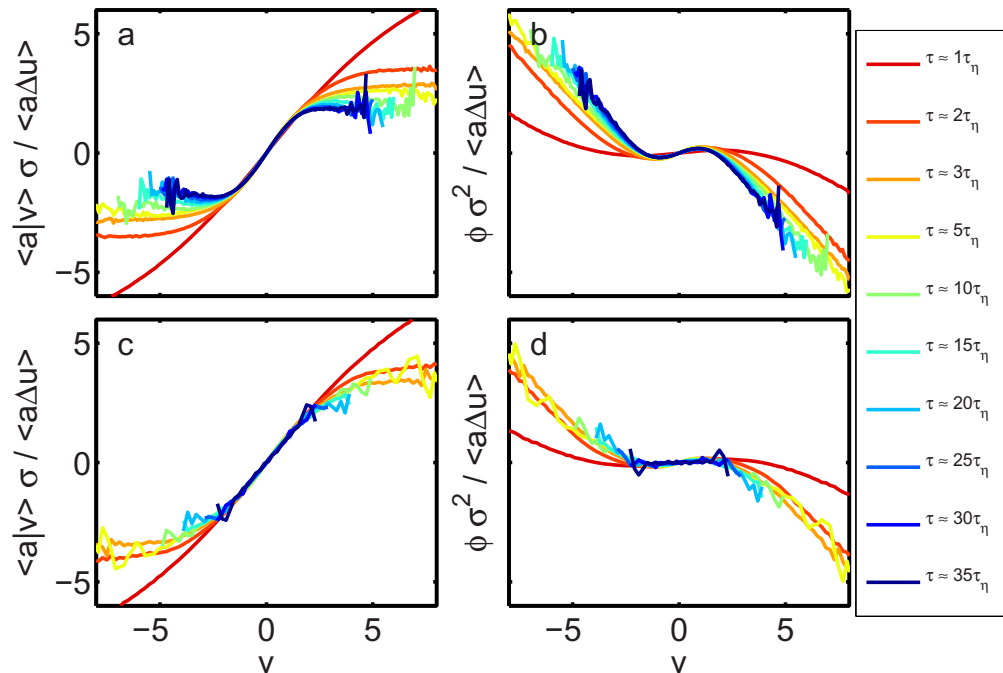
and  $\langle a|v, \tau \rangle$  is the average of the Lagrangian acceleration component  $a = \frac{du}{dt}$  conditioned on the normalized velocity increment. We note that all accelerations are evaluated at time  $t + \tau$ . We emphasize that this equation is an exact kinematic relation, and is not restricted to turbulence. It reveals that the correlations between the acceleration and the velocity increments are the central quantities needed to study Lagrangian intermittency.

Inspection of equation (2) shows that if  $\varphi = 0$ ,  $\partial f / \partial \tau = 0$  and  $f$  is self-similar across scales. But if  $\varphi$  does *not* vanish,  $f$  is not independent of scale, and the increments will be intermittent. Thus, we refer to  $\varphi$  as the *intermittency generating function*.  $\varphi$  depends crucially on the conditional average  $\langle a|v, \tau \rangle$ ; the second term in equation (3) simply keeps the PDF standardized by enforcing a unit standard deviation. But since this conditional average encodes information about the joint statistics of the velocity and the acceleration, equation (2) is unclosed. When we specialize our discussion to turbulent flow,  $\langle a|v, \tau \rangle$  can be interpreted as the key quantity characterizing Lagrangian intermittency. It contains all of the physics of turbulence: the action of pressure gradients, dissipative effects and large-scale forcing, as well as the way that fluid elements sample the (intermittent) Eulerian velocity field [29, 30].

The limiting behavior of  $\langle a|v, \tau \rangle$  in fully developed turbulence is straightforward to specify. For very short times ( $\tau \ll \tau_\eta$ , where  $\tau_\eta$ , the Kolmogorov time scale, is the smallest dynamical time scale in the flow), the velocity increment is proportional to the acceleration and  $\langle a|v, \tau \rangle \approx \frac{\sigma(\tau)}{\tau} v$ . For very long times ( $\tau \gg T_L$ , where  $T_L$ , the integral time scale, is the largest time scale in the flow), the two velocities in the velocity increment are statistically independent and  $\langle a|v, \tau \rangle \approx 0$ . At intermediate times,  $\langle a|v, \tau \rangle$  cannot be linear due to the presence of intermittency: if it were linear, the conditional acceleration would have to take the form  $\langle a|v, \tau \rangle = \frac{\langle a \Delta u \rangle}{\sigma} v$ , which directly implies  $\varphi = 0$ , i.e. non-intermittent velocity increment statistics. Hence, the mere fact that Lagrangian velocity increments are observed to display intermittent statistics dictates that the intermittency generating function must be nonlinear. Aside from this constraint, however, its shape is unknown. Thus,  $\langle a|v, \tau \rangle$  must be measured.

### 3. Results from experiments and direct numerical simulation

We studied  $\langle a|v, \tau \rangle$  both in experiments and in a numerical simulation. We conducted experiments in a von Kármán water flow between counter-rotating discs that has been described in detail previously [31, 32]. We measured Lagrangian trajectories by following the motion of polystyrene tracer particles optically at high speeds and in three dimensions using a predictive particle tracking algorithm [33, 34] that is capable of resolving the Lagrangian acceleration accurately [32]. Here, we report measurements for turbulence at a Taylor-microscale Reynolds number  $R_\lambda$  of 350. We also performed a direct numerical simulation (DNS) of the Navier–Stokes equations with bulk forcing using a standard pseudospectral algorithm and third-order Runge–Kutta time stepping [26]. The simulation ran in a periodic box with  $1024^3$  grid points, giving a Taylor-microscale Reynolds number of  $R_\lambda = 225$ .



**Figure 1.** Conditional acceleration from (a) DNS and (c) experiment, non-dimensionalized by  $\sigma$  and  $\langle a \Delta u \rangle$ . Although the shape changes continuously for  $\tau \lesssim 2\tau_\eta$ , an approximately self-similar evolution is reached at larger scales. Panels (b) and (d) show the non-dimensionalized intermittency generating function  $\phi \sigma^2 / \langle a \Delta u \rangle$  for DNS and experiment, respectively. For the DNS,  $R_\lambda = 225$  and the ratio of the integral time scale and the Kolmogorov time scale was 58. For the experiment,  $R_\lambda = 350$  and the time scale ratio was 87.

Lagrangian trajectories were calculated using a third-order predictor–corrector scheme, and the Eulerian fields were interpolated along the trajectories using tricubic interpolation.

Our measurements of  $\langle a|v, \tau \rangle$  from both the experiments and the simulation are shown in figure 1. For small time lags ( $\tau \lesssim 2\tau_\eta$ ) the conditional acceleration changes smoothly from the asymptotic linear behavior to an approximately sigmoidal shape. But, remarkably, for  $\tau \gtrsim 2\tau_\eta$  (roughly the correlation time of an acceleration component [7, 9]) we find that the curves for different  $\tau$  nearly collapse when non-dimensionalized indicating a self-similar evolution. This is an important observation as it has significant implications on our understanding of Lagrangian intermittency: even though the PDF itself is not self-similar for intermediate time lags, the underlying driving mechanism, the conditional acceleration, is. The collapse is particularly good for small values of the velocity increment. For larger increments, the conditional acceleration levels off and appears to saturate at slightly different values for different values of  $\tau$ . The sigmoidal shape of the conditional acceleration admits an interesting physical interpretation. For small velocity increments, the acceleration depends linearly on the velocity increment: a fluid element will, on average, be driven by an acceleration proportional to its velocity increment, which could be expected from a Taylor expansion at  $v = 0$ . The saturation of the conditional acceleration for larger values, however, implies that beyond a certain value of the velocity increment the expected acceleration does not change anymore. That result in turn suggests

that the ensemble of fluid elements encountering rare, violent velocity fluctuations will all be subject to the same expected acceleration. In this sense, the mechanism that generates the extreme fluctuations observed in the velocity increments can be divided into two separate regimes, both of which are remarkably simple. A quantitative theoretical explanation of the shape of the conditional acceleration is currently lacking, though, and may be regarded as a central challenge for future research. Finally, the observation that this simple description breaks down for the smallest scales can be interpreted as a clear indication of the existence of a Lagrangian dissipative range. Interestingly, a pronounced difference between the dissipative scales and inertial scales has also been reported in the context of the Lagrangian multifractal framework [15].

To quantify the evolution of the PDF across scales, we must study the intermittency generating function  $\varphi$ , which is also shown in figure 1. Non-dimensionalized as in figure 1, the shape of this function can be directly obtained from the conditional acceleration by subtracting the linear contribution with slope one, and thus the above discussion of the conditional acceleration directly translates to the intermittency generating function. To understand how  $\varphi$  gives the intermittent evolution of the PDFs, we must solve equation (2). To do this, we use the method of characteristics. This technique amounts to tracking the evolution of an ‘initial condition’  $v_0$  under the action of the intermittency generating function. The corresponding characteristic curve  $V(\tau, v_0)$  obeys the ordinary differential equation

$$\frac{d}{d\tau} V(\tau, v_0) = [\varphi(v, \tau)]_{v=V(\tau, v_0)}. \quad (4)$$

Along these curves, the PDF evolves according to

$$\frac{d}{d\tau} f(V(\tau, v_0); \tau) = - \left[ \frac{\partial}{\partial v} \varphi(v, \tau) \right]_{v=V(\tau, v_0)} f(V(\tau, v_0); \tau). \quad (5)$$

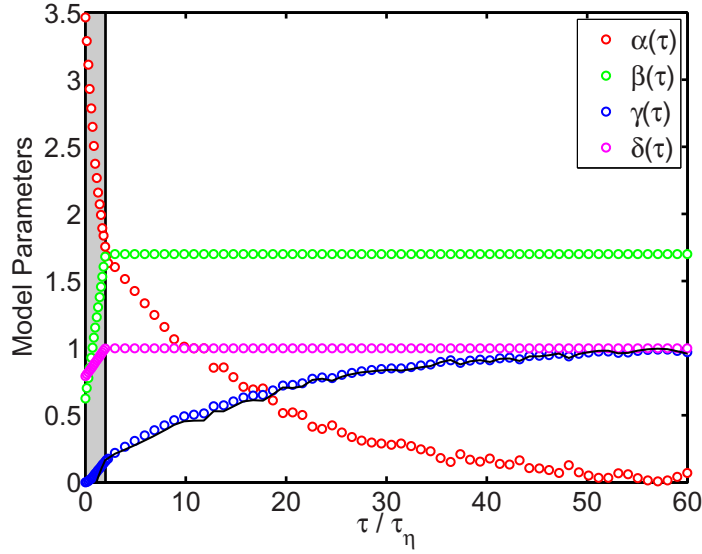
If we take the initial condition to be at large scales where the PDF is nearly Gaussian and proceed toward smaller scales, these equations show that  $\varphi$  squeezes the core of the PDF to smaller increments while simultaneously stretching the tails. Essentially, equation (4) stretches the PDF along the  $v$ -axis while equation (5) adjusts its magnitude so that the PDF remains normalized. Based on our observations above, we know that this squeezing process is self-similar, and yet leads to non-self-similar statistics.

Once we have solved equations (4) and (5), the PDF can be written as

$$f(v; \tau) = \left| \frac{\partial V^{-1}(\tau, v)}{\partial v} \right| f(V^{-1}(\tau, v); \tau_0), \quad (6)$$

where  $V^{-1}$  is the inverse of  $V$ . Equation (6) shows that the method of characteristics naturally introduces the mapping of an initial PDF  $f(v_0; \tau_0)$  to the PDF at some other scale  $\tau \neq \tau_0$ , similar to the mapping closure technique [35]. If we choose  $\tau_0$  to be a large time lag, the initial PDF is well approximated by a Gaussian.

At this point, we have reached our main result: that intermittency can be captured by a single function that is only weakly scale-dependent. In the following section, we discuss a secondary question: how can we use this framework to extract quantitative information about Lagrangian intermittency? As an example, we construct a model for the velocity increment PDFs rooted in this framework.



**Figure 2.** Model parameters, obtained by fitting to the DNS results. For  $\tau > 2\tau_\eta$   $\alpha$  and  $\gamma$  are the only  $\tau$ -dependent parameters, and are linked by the unit standard deviation condition. The black line shows an analytical approximation of  $\gamma$  based on the value of  $\alpha$ , demonstrating that there is effectively a single free parameter function in our model. For small  $\tau$ , indicated by the gray region, we use a slightly more complex model (see text).

#### 4. Modeling the velocity increment PDFs

It turns out that equation (6) cannot simply be solved analytically for the general nonlinear shape of  $\varphi$  observed in our experimental and numerical data. Thus, to formulate an analytically tractable model for the increment PDF, we use the observation that  $\varphi$  is a smooth blending between two linear regimes for intermediate to large time lags ( $\tau \gtrsim 2\tau_\eta$ ). In the following, we will refer to this fact simply as ‘piecewise linearity’. The piecewise linearity of the intermittency generating function is a direct consequence of the two distinct regimes observed for the conditional acceleration.

It can be shown that piecewise linearity of the intermittency generating function  $\varphi$  implies piecewise linearity of the inverse of the characteristic curve  $V^{-1}$ . We thus make the simple ansatz

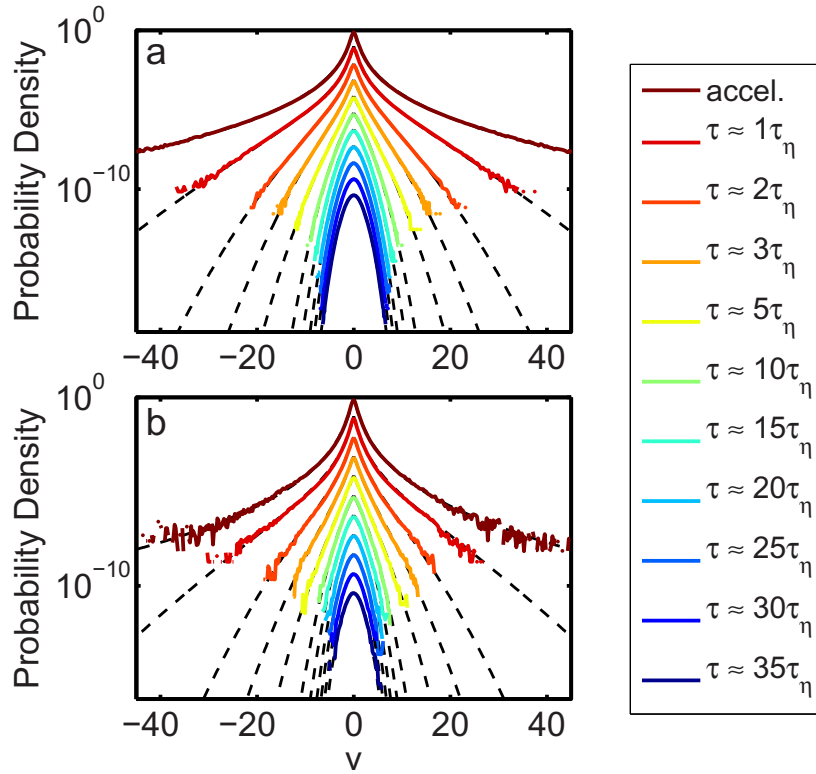
$$V^{-1}(\tau, v) = \frac{\alpha(\tau)v}{1 + |v|/\beta} + \gamma(\tau)v, \quad (7)$$

where  $\beta$  is the scale at which  $V^{-1}$  transitions between two linear functions, one with slope  $(\alpha + \gamma)$  for small  $v$  and one with slope  $\gamma$  for large  $v$ .  $\beta$  can be estimated as the scale seen in figure 1 at which the conditional acceleration saturates; for the DNS data, we choose  $\beta = 1.7$ . To fit  $\alpha$  and  $\gamma$ , we first note that equation (6) can be solved by integration, yielding

$$V^{-1}(\tau, v) = F^{-1}(F(v; \tau); \tau_0), \quad (8)$$

where  $F(v; \tau) = \int_{-\infty}^v dv' f(v'; \tau)$  is the cumulative distribution function, and we have assumed that  $\partial V^{-1}/\partial v > 0$ . We note in passing that equation (8) is a rather general relation between two PDFs and might also be useful for applications other than the one presented here.





**Figure 3.** Acceleration and velocity increment PDFs measured from (a) DNS and (b) experiment (solid lines), together with the predictions of our model (dashed lines). PDFs for different scales have been offset vertically for clarity. Remarkable agreement is found for all scales.

For (large-scale) Gaussian initial conditions, we have  $F(v; \tau_0) = \frac{1}{2}[1 + \text{erf}(v/\sqrt{2})]$ . With this relation,  $V^{-1}$  can be directly obtained from the measured PDFs. For each  $\tau$ , we then fit values of  $\alpha$  and  $\gamma$ , and the result is shown in figure 2. To reduce the number of free parameters by one, we note that  $\alpha$  and  $\gamma$  cannot be chosen independently; they are rather related by the unit standard deviation condition, as detailed below.

If we also want to capture the statistics of the acceleration (the limit of the increment as  $\tau \rightarrow 0$ ), we must generalize our simple model ansatz and include the non-trivial behavior of the conditional acceleration in the dissipative range. To this end we replace the denominator in equation (7) by  $1 + [|v|/\beta]^\delta$  and impose  $\gamma(0) = 0$ , which leads to the stretched exponential tails seen in experiments [6, 8]. From the DNS data, we obtain  $\alpha \approx 3.5$ ,  $\beta \approx 0.6$  and  $\delta \approx 0.8$ . In the dissipative range, the PDF must relax from the acceleration PDF to the velocity increment PDF at finite  $\tau$ . We model this shape change by a linear variation of  $\beta$  and  $\delta$  to their larger-scale values.  $\alpha$  and  $\gamma$  remain constrained by the unit standard deviation condition. The full model parameters, capturing both the acceleration and the velocity increments, are shown in figure 2. Apart from delivering a good parametrization of the velocity increment PDFs, this modeling approach once more shows that the description of the Lagrangian dissipative range is more challenging than the description of the inertial range.

To further elucidate the interrelation of  $\alpha$  and  $\gamma$ , the requirement of unit standard deviation for the PDFs imposes the condition

$$\int dv v^2 f(v; \tau) = \int dV^{-1} v(V^{-1})^2 f(V^{-1}; \tau_0) = 1. \quad (9)$$

Although this equation cannot be solved analytically for the assumed form of  $V^{-1}$  and a Gaussian large-scale PDF, an approximate relation can be derived. By linearizing  $V^{-1}$  about the crossover scale  $\beta$ , we evaluate this condition and obtain

$$\frac{\alpha^2 \beta^2 \delta^2 + 16 - 8\sqrt{\frac{2}{\pi}} \alpha \beta \delta}{(\alpha \delta - 2\alpha - 4\gamma)^2} = 1, \quad (10)$$

which allows us to express  $\alpha$  in terms of  $\gamma$  or vice versa. This approximate relation, together with the other model parameters, is shown in figure 2. Thus, our model has effectively only one single free parameter function for the intermediate to large scales (where  $\delta = 1$  and  $\beta = 1.7$  are fixed), and is slightly more complex for the smallest time lags.

The PDFs based on this model are shown in figure 3 along with measurements of the PDFs from experiment and DNS. The agreement between the model results and the data is very encouraging. The piecewise linearity of the model (equation (7)) shows that the intermittent PDFs for intermediate scales can be thought of as a smoothly matched combination of two Gaussian PDFs with differing standard deviations. One immediate consequence is that the tail behavior of the velocity increment PDFs is Gaussian. This can be seen from the linear behavior of equation (7) for  $|v| \gg \beta$ , which inserted into relation (6) with a Gaussian PDF for large time lags implies a Gaussian tail behavior. This is a prediction of the model that can be checked carefully in future comparisons with experiments and DNS.

## 5. Summary and conclusions

In summary, we have described a new way of characterizing intermittency in turbulence without making any initial phenomenological assumptions about the nature of the flow field. We introduced an exact, although unclosed, kinematic equation for the evolution of the PDF of the velocity increments in scale, and showed that a single function, the intermittency generating function, controls the shape deformation of the PDF. This function in turn depends crucially on the statistical dependence of the acceleration on the velocity history of the fluid element, which can be interpreted as a Lagrangian analogue of Kolmogorov's refined similarity hypothesis [3]: the refined similarity hypothesis takes into account statistical correlations of dissipation and velocity to describe Eulerian intermittency, whereas the presented approach shows that the understanding of correlations between acceleration and velocity increments is necessary to capture Lagrangian intermittency. Thus, our results are consistent with the expectation that intermittency results from the direct coupling of large and small scales, but furthermore show precisely how this coupling arises, namely in terms of the conditional acceleration. Measurements of this quantity in experiment and DNS showed a distinct shape transition in the dissipative range followed by an approximately self-similar evolution for moderate to large time lags. Thus, although the evolution of the velocity increment is not self-similar in this range

of scales, the underlying mechanism nearly is. This is both a surprising and significant finding, whose physical origin remains to be understood.

Within this self-similar regime, the conditional acceleration has a particularly simple piecewise linear dependence on the velocity increment. Using this observation, we introduced a model for the velocity PDF that captures its shape well across all scales. Further work on intermittency should focus on developing a theoretical understanding of the form of the intermittency generating function.

Moreover, it would be of considerable interest to relate our findings to other modeling approaches like the multifractal phenomenology. Intermittent velocity increment PDFs can be obtained by a superposition of Gaussian PDFs of varying standard deviations [15, 36], where the distribution of standard deviations has been derived within the multifractal framework in [15]. Combining this ansatz with the PDF equation presented in this paper shows that the evolution of the multifractal distribution across scales is governed by the conditional acceleration. Further pursuit of this observation together with the question of what insight the multifractal framework could bring to our understanding of the peculiar shape of the intermittency generating function is work in progress.

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